ON A THEOREM OF BOMBIERI, FRIEDLANDER AND IWANIEC

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Abstract. In this article, we show to which extent one can improve a theorem of Bombieri, Friedlander and Iwaniec by using Hooley's variant of the divisor switching technique. We also give an application of the theorem in question, which is a Bombieri-Vinogradov type theorem for the Tichmarsh divisor problem in arithmetic progressions.

1. Introduction

The Bombieri-Vinogradov theorem implies that on average over $q \leq x^{1/2-o(1)}$, the primes less than x are equidistributed in the residue classes $a \bmod q$, with $(a,q)=1$. Specifically, we have for any $A>0$ that

$$\sum_{q \leq Q} \max_{a:(a,q)=1} \left| \psi(x;q,a) - \frac{x}{\phi(q)} \right| \ll \frac{x}{(\log x)^A}, \tag{1}$$

where $Q = x^{1/2}/(\log x)^{A+5}$. One could ask if (1) still holds if we take $Q = x^{\theta}$, with $\theta > \frac{1}{2}$. This would be a major achievement, since it would imply bounded gaps between primes [12], that is

$$\liminf_n (p_{n+1} - p_n) < \infty.$$

The Elliot-Halberstam conjecture stipulates that we can take θ to be any real number less than 1. This conjecture is however very far from reach.

One way to get past the barrier of $Q = x^{1/2-o(1)}$ is to relax the condition on a. Indeed, in concrete problems, one often only needs the bound (1) for a fixed value of a. Sometimes, even the absolute values are not necessary. These variants were studied very closely in a series of groundbreaking articles by Fouvry & Iwaniec ([8], [9]), Fouvry ([5], [6], [7]), and Bombieri, Friedlander & Iwaniec ([1], [2], [3]). We will list the results of these authors by increasing order of uniformity.

By fixing a, one can go up to $Q = x^{\frac{1}{2} + \frac{1}{(\log\log x)^B}}$.

Theorem 1.1 (Bombieri, Friedlander and Iwaniec [2]). *Let $a \neq 0$, $x \geq y \geq 3$, and $Q^2 \leq xy$. We then have*

$$\sum_{\substack{Q \leq q < 2Q \\ (q,a)=1}} \left| \psi(x;q,a) - \frac{x}{\phi(q)} \right| \ll x \left(\frac{\log y}{\log x} \right)^2 (\log\log x)^B.$$

The best known result was obtained shortly afterwards by the same authors, and shows that one can go up to $Q = x^{\frac{1}{2}+o(1)}$, whatever the nature of the $o(1)$ is.

Theorem 1.2 (Bombieri, Friedlander, Iwaniec [3]). *Let $a \neq 0$ be an integer and $A > 0$, $2 \leq Q \leq x^{3/4}$ be reals. Let $\mathcal{Q}$ be the set of all integers q, prime to a, from an interval $Q' < q \leq Q$. Then*

$$\sum_{q\in\mathcal{Q}} \left|\pi(x;q,a) - \frac{\pi(x)}{\phi(q)}\right| \leq \left\{K\left(\theta-\frac{1}{2}\right)^2 \frac{x}{\log x} + O_A\left(\frac{x(\log\log x)^2}{(\log x)^3}\right)\right\}\sum_{q\in\mathcal{Q}}\frac{1}{\phi(q)} + O_{a,A}\left(\frac{x}{(\log x)^A}\right),$$

where $\theta := \frac{\log Q}{\log x}$ and K is absolute.

Replacing the absolute values by a certain weight (see [1] for the definition of "well factorable"), we can take $Q = x^{4/7-\epsilon}$.

Theorem 1.3 (Bombieri, Friedlander and Iwaniec [1]). *Let $a \neq 0$, $\epsilon > 0$ and $Q = x^{4/7-\epsilon}$. For any well factorable function $\lambda(q)$ of level Q and any $A > 0$ we have*

$$\sum_{(q,a)=1} \lambda(q)\left(\psi(x;q,a) - \frac{x}{\phi(q)}\right) \ll \frac{x}{(\log x)^A}. \tag{2}$$

Theorem 1.3 is an improvement of a result of Fouvry & Iwaniec [9], which showed that (2) holds with $\lambda(q)$ of level $Q = x^{9/17-\epsilon}$.

If we remove the weight $\lambda(q)$, we can take $Q = x/(\log x)^B$, which is even further than in the Elliot-Halberstam conjecture. This result was obtained independently by Fouvry [7] and Bombieri, Friedlander & Iwaniec [1] (in stronger form).

Theorem 1.4 (Bombieri, Friedlander and Iwaniec [1]). *Let $a \neq 0$, $\lambda < \frac{1}{10}$ and $R < x^\lambda$. For any $A > 0$ there exists $B = B(A)$ such that provided $QR < x/(\log x)^B$ we have*

$$\sum_{\substack{r\leq R\\(r,a)=1}} \left|\sum_{\substack{q\leq Q\\(q,a)=1}} \left(\psi(x;qr,a) - \Lambda(a) - \frac{x}{\phi(qr)}\right)\right| \ll_{a,A,\lambda} \frac{x}{(\log x)^A}. \tag{3}$$

Remark 1.5. We subtracted $\Lambda(a)$ from $\psi(x;qr,a)$ in (3) because the arithmetic progression $a \bmod qr$ contains the prime power p^e for all values of qr if $a = p^e$. This induces a negligible error term in (3) (for $B > A$).

In this article we focus on Theorem 1.4. We show in Corollary 3.2 that for any $A > 0$,

- If $a = \pm 1$, then Theorem 1.4 holds if $B(A) > A$, and is false if $B(A) = A$.
- If $a = \pm p^e$, then Theorem 1.4 holds if $B(A) = A$, and is false if $B(A) < A$.
- If a has more than two prime factors, then Theorem 1.4 holds if $B(A) > \frac{538}{743}A$.

One of the applications of Theorem 1.4 and of Fouvry's result [7] is the best known estimate for the Titchmarsh divisor problem. We will show that Theorem 1.4 yields a generalization of this result, that is a Bombieri-Vinogradov type result for the Titchmarsh divisor problem in arithmetic progressions, up to level $Q = x^{1/10-\epsilon}$.

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3. Statement of results

For an integer $r \geq 1$, we will use the notation

$$r' := \prod_{p|r} p.$$

Here is our main result.

Theorem 3.1. *Fix an integer $a \neq 0$ and two positive real numbers $\lambda < \frac{1}{10}$ and A. We have for $R = R(x) \leq x^{\lambda}$ and $M = M(x) \leq (\log x)^A$ that*

$$\sum_{\substack{\frac{R}{2} < r \leq R \\ (r,a)=1}} \left| \sum_{\substack{q \leq \frac{x}{rM} \\ (q,a)=1}} \left(\psi(x; qr, a) - \Lambda(a) - \frac{x}{\phi(qr)} \right) - \frac{\phi(a)}{a} \frac{x}{rM} \mu(a, r, M) \right| \ll_{a,A,\epsilon,\lambda} \frac{x}{M^{\frac{743}{538}-\epsilon}},$$

where the "average" is given by

$$\mu(a, r, M) := \begin{cases} -\frac{1}{2} \log M - C_5(r) & \text{if } a = \pm 1 \\ -\frac{1}{2} \log p & \text{if } a = \pm p^e \\ 0 & \text{otherwise,} \end{cases}$$

with

$$C_5(r) := \frac{1}{2} \left(\log 2\pi + 1 + \gamma + \sum_p \frac{\log p}{p(p-1)} + \sum_{p|r} \frac{\log p}{p} \right).$$

We also have the following similar result:

$$\sum_{\substack{r \leq R \\ (r,a)=1}} \left| \sum_{\substack{q \leq \frac{x}{RM} \\ (q,a)=1}} \left(\psi(x; qr, a) - \Lambda(a) - \frac{x}{\phi(qr)} \right) - \frac{\phi(a)}{a} \frac{x}{RM} \mu(a, r, RM/r) \right| \ll_{a,A,\epsilon,\lambda} \frac{x}{M^{\frac{743}{538}-\epsilon}}.$$

As a corollary, we get a more precise form of Theorem 1.4.

Corollary 3.2. *Fix an integer $a \neq 0$ and two positive real numbers $\lambda < \frac{1}{10}$ and A. We have for $R = R(x) \leq x^{\lambda}$ and $M = M(x) \leq (\log x)^A$ that*

$$\sum_{\substack{r \leq R \\ (r,a)=1}} \left| \sum_{\substack{q \leq \frac{x}{RM} \\ (q,a)=1}} \left(\psi(x; qr, a) - \Lambda(a) - \frac{x}{\phi(qr)} \right) \right| = \left(\frac{\phi(a)}{a} \right)^2 \frac{x}{M} \nu(a, M) + O_{a,A,\epsilon,\lambda} \left(\frac{x}{M^{\frac{743}{538}-\epsilon}} \right),$$

where

$$\nu(a,M) := \begin{cases} \frac{1}{2}\log M + C_6 + O\left(\frac{\log(RM)}{R}\right) & \text{if } a = \pm 1 \\ \frac{1}{2}\log p + O\left(\frac{1}{R}\right) & \text{if } a = \pm p^e \\ 0 & \text{otherwise,} \end{cases}$$

with

$$C_6 := C_5(1) + \frac{1}{2} + \frac{1}{2}\sum_p \frac{\log p}{p^2}.$$

Remark 3.3. If a has at most 1 prime factor, then for M and R both tending to infinity we have that

$$\nu(a,M) \sim \begin{cases} \frac{1}{2}\log M & \text{if } a = \pm 1 \\ \frac{1}{2}\log p & \text{if } a = \pm p^e. \end{cases}$$

(If R is bounded, then we should multiply by $\frac{a}{\phi(a)}\frac{\#\{r\leq R:(r,a)=1\}}{R}$ in the case $a = \pm p^e$, and by $\frac{\lfloor R\rfloor}{R}$ in the case $a = \pm 1$.)

Another corollary of our results (which actually follows from Theorem 1.4) is a Bombieri-Vinogradov type result for the Titchmarsh divisor problem in arithmetic progressions. We use the following notation for the divisor function: $\tau(n) := \sum_{d|n} 1$.

Theorem 3.4. *Fix an integer $a \neq 0$ and let $\lambda < \frac{1}{10}$ and A be two fixed positive real numbers. We have for $Q \leq x^\lambda$ that*

$$\sum_{q\leq Q}\left|\sum_{|a|/q<m\leq x/q} \Lambda(qm+a)\tau(m) - M.T.\right| \ll_{a,A,\lambda} \frac{x}{(\log x)^A}, \tag{4}$$

where the main term is

$$M.T. := \frac{x}{q}\left(C_1(a,q)\log x + 2C_2(a,q) + C_1(a,q)\log\left(\frac{(q')^2}{eq}\right)\right),$$

with $C_1(a,q)$ and $C_2(a,q)$ defined as in section 4.

A version of Theorem 3.4 was obtained independently by Felix [4], who also showed how to apply this result to a question related to Artin's primitive root conjecture. Using Theorem 3.4, one can give a slight improvement of Theorem 1.5 of [4], that is replace $O(\log\log x)$ by $c\log\log x + O(1)$, for some constant c.

Taking $Q = (\log x)^C$ in Theorem 3.4, we obtain a "Siegel-Walfisz theorem" for the Titchmarsh divisor problem, and one could ask if this is sufficient to give the bound (4) for $Q = x^{1/2}/(\log x)^B$, since it is known that the Bombieri-Vinogradov theorem holds with fairly general sequences satisfying the Siegel-Walfisz condition. If this is true, then it would yield the following improvement of a dyadic version of Theorem 1.4.

Proposition 3.5. *Fix an integer $a \neq 0$, a real number $A > 0$ and let $R = R(x) \leq x^{1/2}/(\log x)^{3A+5}$. Assume that (4) holds for $Q = R(x)$. Then for $L := (\log x)^{A+3}$ we have*

$$\sum_{\substack{\frac{R}{2}<r\leq R \\ (r,a)=1}}\left|\sum_{\substack{q\leq \frac{x}{RL} \\ (q,a)=1}}\left(\psi(x;qr,a) - \Lambda(a) - \frac{x}{\phi(qr)}\right)\right| \ll_{a,A} \frac{x}{(\log x)^A}. \tag{5}$$

4. Notation

We will denote by γ the Euler-Mascheroni constant. We also define the following constants:

$$C_1(a,r) := \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a} \left(1 - \frac{p}{p^2-p+1}\right) \prod_{p|r} \left(1 + \frac{p-1}{p^2-p+1}\right),$$

$$C_2(a,r) := C_1(a,r) \left(\gamma - \sum_p \frac{\log p}{p^2-p+1} + \sum_{p|a} \frac{p^2 \log p}{(p-1)(p^2-p+1)} - \sum_{p|r} \frac{(p-1)p\log p}{p^2-p+1}\right),$$

$$C_3(a,r) := C_2(a,r) - C_1(a,r),$$

$$C_5(r) := \frac{1}{2}\left(\log 2\pi + 1 + \gamma + \sum_p \frac{\log p}{p(p-1)} + \sum_{p|r} \frac{\log p}{p}\right).$$

Moreover, for $i = 1,2,3$,

$$C_i(a) := C_i(a,1),$$

and

$$C_5 := C_5(1).$$

We denote by $\omega(n)$ the number of prime factors of n.

5. Preliminary lemmas

We start with some elementary estimates.

Lemma 5.1. *Let f be a multiplicative function and g an additive function, that is for $(m,n)=1$, $f(mn) = f(m)f(n)$ and $g(mn) = g(m)+g(n)$ (in particular, $f(1) = 1$ and $g(1) = 0$). Then for a squarefree integer r we have that*

$$\sum_{d|r} f(d)g(d) = \prod_{p'|r}(1+f(p')) \sum_{p|r} \frac{g(p)f(p)}{1+f(p)}.$$

Proof. We write

$$\sum_{d|r} f(d)g(d) = \sum_{d|r} f(d) \sum_{p|r} g(p) = \sum_{p|r} g(p) \sum_{\substack{d|r:\\ p|d}} f(d) = \sum_{p|r} g(p) \sum_{d|\frac{r}{p}} f(p)f(d)$$

$$= \sum_{p|r} g(p)f(p) \prod_{p'|\frac{r}{p}} (1+f(p')) = \sum_{p|r} \frac{g(p)f(p)}{1+f(p)} \prod_{p'|r}(1+f(p')).$$

□

Lemma 5.2. *Let a and r be coprime integers, with r squarefree. We have for $i = 1,2$ that*

$$\frac{C_i(a,r)}{r} = \sum_{d|r} \mu(d) C_i(ad). \tag{6}$$

Proof. By the definition of $C_1(a)$, we have

$$\sum_{d|r}\mu(d)C_1(ad) = C_1(a)\prod_{p|r}\left(1-\left(1-\frac{p}{p^2-p+1}\right)\right) = \frac{C_1(a,r)}{r}.$$

Moreover, by defining the multiplicative function $f(d) := \frac{\zeta(6)}{\zeta(2)\zeta(3)}\mu(d)C_1(d)$ we have

$$\begin{aligned}\sum_{d|r}\mu(d)C_2(ad) &= C_1(a)\sum_{d|r}f(d)\left(\gamma-\sum_p\frac{\log p}{p^2-p+1}+\sum_{p|a}\frac{p^2\log p}{(p-1)(p^2-p+1)}\right)\\ &\quad + C_1(a)\sum_{d|r}f(d)\sum_{p|d}\frac{p^2\log p}{(p-1)(p^2-p+1)}\\ &= C_2(a)\prod_{p|r}\frac{p}{p^2-p+1}+C_1(a)\sum_{d|r}f(d)\sum_{p|d}\frac{p^2\log p}{(p-1)(p^2-p+1)}.\end{aligned}$$

Applying Lemma 5.1, we get that this is

$$\begin{aligned}&= C_2(a)\prod_{p|r}\frac{p}{p^2-p+1}+C_1(a)\prod_{p'|r}(1+f(p'))\sum_{p|r}\frac{p^2\log p}{(p-1)(p^2-p+1)}\frac{f(p)}{1+f(p)}\\ &= C_2(a)\prod_{p|r}\frac{p}{p^2-p+1}-C_1(a)\prod_{p'|r}\frac{p'}{(p')^2-p'+1}\sum_{p|r}\frac{(p-1)p\log p}{p^2-p+1}\\ &= C_1(a)\prod_{p|r}\frac{p}{p^2-p+1}\left(\gamma-\sum_p\frac{\log p}{p^2-p+1}+\sum_{p|a}\frac{p^2\log p}{(p-1)(p^2-p+1)}-\sum_{p|r}\frac{(p-1)p\log p}{p^2-p+1}\right)\\ &= \frac{C_2(a,r)}{r}.\end{aligned}$$

□

Lemma 5.3. *Fix $r>0$ and $a\neq 0$ two coprime integers. We have*

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{n}{\phi(n)} = C_1(a)M+O(2^{\omega(a)}\log M),$$

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{1}{\phi(n)} = C_1(a)\log M+C_2(a)+O\left(2^{\omega(a)}\frac{\log M}{M}\right),$$

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{rn}{\phi(rn)} = C_1(a,r)M+O\left(3^{\omega(ar)}\log(r'M)\right),$$

$$\sum_{\substack{n\le M\\(n,a)=1}}\frac{1}{\phi(rn)} = \frac{C_1(a,r)}{r}\log(r'M)+\frac{C_2(a,r)}{r}+O\left(3^{\omega(ar)}\frac{\log(r'M)}{rM}\right).$$

Proof. For the first two estimates, see [10] or [11]. We now sketch a proof the last estimate. First we assume that r is squarefree, since if it is not we can write

$$\frac{1}{\phi(rn)} = \frac{r'}{r\phi(r'n)}.$$

Then, we use the identity

$$\sum_{\substack{d|r \\ (d,n)=1}} \mu(d) = \begin{cases} 1 & \text{if } r \mid n \\ 0 & \text{else} \end{cases}$$

to write

$$\sum_{\substack{n\leq M \\ (n,a)=1}} \frac{1}{\phi(rn)} = \sum_{d|r} \mu(d) \sum_{\substack{n\leq rM \\ (n,ad)=1}} \frac{1}{\phi(n)}.$$

Now, substituting in the $r=1$ estimate, we get that

$$\sum_{\substack{n\leq M \\ (n,a)=1}} \frac{1}{\phi(rn)} = \log(rM) \sum_{d|r} \mu(d) C_1(ad) + \sum_{d|r} \mu(d) C_2(ad) + O\left(3^{\omega(ar)} \frac{\log(rM)}{rM}\right).$$

The result follows by Lemma 5.2. $\square$

Lemma 5.4. *Fix $r>0$ and $a \neq 0$ two coprime integers.*

If $\omega(a) \geq 1$,

$$\sum_{\substack{n\leq M \\ (n,a)=1}} \frac{1}{\phi(nr)} \left(1 - \frac{n}{M}\right) = \frac{C_1(a,r)}{r} \log(r'M) + \frac{C_3(a,r)}{r} + \frac{\phi(a)}{a} \frac{\Lambda(a)}{2rM} + E(a,r,M).$$

If $a = \pm 1$,

$$\sum_{\substack{n\leq M \\ (n,a)=1}} \frac{1}{\phi(nr)} \left(1 - \frac{n}{M}\right) = \frac{C_1(1,r)}{r} \log(r')M + \frac{C_3(1,r)}{r} + \frac{\log(r'M)}{2rM} + \frac{C_5}{rM} + E(a,r,M).$$

The error term satisfies

$$E(a,r,M) \ll \frac{\prod_{p|ar}\left(1+\frac{1}{p^\delta}\right)}{rM} \left(\frac{a'}{M}\right)^{\frac{205}{538}-\epsilon},$$

for some $\delta > 0$.

Proof. For the proof in the case $r=1$, we refer the reader to Lemma 6.9 of [10]. In the proof, we replace (40) by the bound

$$\mathfrak{S}_{a_M}(s+1) \ll a_M^{-1-\sigma} \prod_{p|a_M} \left(1 + \frac{1}{p^\delta}\right),$$

which will yield the improved error term

$$E(a,1,M) \ll \frac{\prod_{p|a}\left(1+\frac{1}{p^\delta}\right)}{M} \left(\frac{a'}{M}\right)^{\frac{205}{538}-\epsilon}.$$

Note that the exponent $\frac{205}{538}$ comes from Huxley's subconvexity bound on $\zeta(s)$ [14].

For the general case, we proceed as in the preceding lemma. We can again assume that r is squarefree, and write

$$\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(nr)}\left(1-\frac{n}{M}\right) = \sum_{d|r}\mu(d) \sum_{\substack{n\le rM\\(n,ad)=1}} \frac{1}{\phi(n)}\left(1-\frac{n}{rM}\right),$$

in which we substitute the $r=1$ estimate. If $\omega(a)\ge 2$, then $\omega(ad)\ge 2$ for all $d\mid r$, so we get

$$\begin{aligned}\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(rn)}\left(1-\frac{n}{M}\right) &= \sum_{d|r}\mu(d)\left(C_1(ad)\log(rM)+C_3(ad)+E(ad,1,rM)\right)\\ &= C_1(a,r)\log(rM)+C_3(a,r)+E(a,r,M)\end{aligned}$$

by Lemma 5.2. Here,

$$\begin{aligned}E(a,r,M) &\ll \sum_{d|r} \frac{\prod_{p|ad}\left(1+\frac{1}{p^\delta}\right)}{rM}\left(\frac{a'd}{rM}\right)^{\frac{205}{538}-\epsilon}\\ &= \frac{\prod_{p|a}\left(1+\frac{1}{p^\delta}\right)}{rM}\left(\frac{a'}{rM}\right)^{\frac{205}{538}-\epsilon}\sum_{d|r} d^{\frac{205}{538}-\epsilon}\prod_{p|d}\left(1+\frac{1}{p^\delta}\right)\\ &= \frac{\prod_{p|a}\left(1+\frac{1}{p^\delta}\right)}{rM}\left(\frac{a'}{rM}\right)^{\frac{205}{538}-\epsilon}\prod_{p|r}\left(1+p^{\frac{205}{538}-\epsilon}\left(1+\frac{1}{p^\delta}\right)\right)\\ &\ll \frac{\prod_{p|ar}\left(1+\frac{1}{p^\delta}\right)}{rM}\left(\frac{a'}{M}\right)^{\frac{205}{538}-\epsilon},\end{aligned}$$

where we might have to change the value of $\delta>0$.

If $\omega(a)=1$, then $\omega(ad)\ge 1$ for all $d\mid r$, so we get

$$\begin{aligned}\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(rn)}\left(1-\frac{n}{M}\right) &= \sum_{d|r}\mu(d)\left(C_1(ad)\log(rM)+C_3(ad)+\frac{\phi(ad)}{ad}\frac{\Lambda(ad)}{2rM}+E(ad,1,rM)\right)\\ &= \sum_{d|r}\mu(d)\left(C_1(ad)\log(rM)+C_3(ad)\right)+\frac{\phi(a)}{a}\frac{\Lambda(a)}{2rM}+E(a,r,M)\\ &= C_1(a,r)\log(rM)+C_3(a,r)+\frac{\phi(a)}{a}\frac{\Lambda(a)}{2rM}+E(a,r,M).\end{aligned}$$

If $a=\pm1$, then we get

$$\sum_{\substack{n\le M\\(n,a)=1}} \frac{1}{\phi(rn)}\left(1-\frac{n}{M}\right) = \sum_{d|r}\mu(d)(C_1(ad)\log(rM)+C_3(ad)+E(ad,1,rM))$$
$$-\sum_{p|r}\frac{\phi(p)}{p}\frac{\Lambda(p)}{2rM}+\frac{\log(rM)}{2rM}+\frac{C_5}{rM}$$
$$=C_1(a,r)\log(rM)+C_2(a,r)+\frac{\log M}{2rM}+\frac{C_5(r)}{rM}+E(a,r,M).$$
□

6. Further results and proofs

Proposition 6.1. *Fix two positive real numbers $\lambda < \frac{1}{10}$ and D. let $M = M(r,x)$ be an integer such that $1 \le M(r,x) \le (\log x)^D$. Then for $R = R(x) \le x^{\lambda}$ we have*

$$\sum_{\substack{R/2<r\le R\\(r,a)=1}}\Bigg|\sum_{\substack{q\le\frac{x}{rM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)$$
$$-x\left(\frac{C_1(a,r)}{r}\log(r'M)+\frac{C_3(a,r)}{r}-\sum_{\substack{s\le M\\(s,a)=1}}\frac{1}{\phi(rs)}\left(1-\frac{s}{M}\right)\right)\Bigg| = O_{a,A,D,\lambda}\left(\frac{x}{\log^A x}\right). \quad (7)$$

We can remove the condition of M being an integer at the cost of adding the error term $O\left(x\frac{\log\log M}{M^2}\right)$.

Proof. The proof follows closely that of Proposition 7.1 of [10]. We start by splitting the sum over q as follows:

$$\sum_{\substack{q\le\frac{x}{rM}\\(q,a)=1}} = \sum_{\substack{q\le\frac{x}{RL}\\(q,a)=1}} + \sum_{\substack{\frac{x}{RL}<q\le\frac{x}{r}\\(q,a)=1}} - \sum_{\substack{\frac{x}{rM}<q\le\frac{x}{r}\\(q,a)=1}}.$$

We use Theorem 1.4 to bound the first of these sums by taking $L := (\log x)^{A+B+D+4}$, with $B = B(A)$ coming from this theorem:

$$\sum_{\substack{R/2<r\le R\\(r,a)=1}}\Bigg|\sum_{\substack{q\le\frac{x}{RL}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\Bigg| \ll_{a,A,D,\lambda} \frac{x}{(\log x)^A}.$$

We study the two remaining sums in the same way, by writing

$$\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right) = \sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\sum_{\substack{|a|<n\le x\\n\equiv a \bmod qr}}\Lambda(n) - x\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\frac{1}{\phi(qr)},$$

where we will take $P \le 2L$ to be either M or $\frac{RL}{r}$. The last term on the right is easily treated using Lemma 5.3. As for the first term, we can remove the prime powers at the cost of a

negligible error term, and end up with the following sum:

$$\sum_{\substack{\frac{x}{rP}<q\le\frac{x}{r}\\(q,a)=1}}\ \sum_{\substack{|a|<p\le x\\p\equiv a \bmod qr}} \log p.$$

We will now use Hooley's variant of the divisor switching technique (see [13]). Writing $p = a + qrs$, we see that we should sum over s rather than over q, since the bound $\frac{x}{rP} < q$ forces s to be very small. We get that the sum is, up to an error $\ll (\log x)^2$, equal to

$$\begin{aligned}\sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}}\ \sum_{\substack{\frac{sx}{P}+a\le p\le x\\p\equiv a \bmod sr}} \log p &= \sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}} \left(\theta(x;sr,a)-\theta\left(\frac{sx}{P}+a;sr,a\right)\right)\\ &= \sum_{\substack{1\le s<P-\frac{aP}{x}\\(s,a)=1}} \frac{x}{\phi(sr)}\left(1-\frac{s}{P}\right)+E(r,a),\end{aligned}$$

where, by the Bombieri-Vinogradov theorem,

$$\begin{aligned}\sum_{\substack{R/2<r\le R\\(r,a)=1}} |E(r,a)| &\le \sum_{\substack{s\le 2L\\(s,a)=1}}\ \sum_{\substack{r\le R\\(r,a)=1}} \max_{y\le x}\left|\theta(y;sr,a)-\frac{y}{\phi(sr)}\right| + O_{a,A}\left(\frac{x}{(\log x)^A}\right)\\ &\le 2L \sum_{\substack{q\le 2RL\\(q,a)=1}} \max_{y\le x}\left|\theta(y;q,a)-\frac{y}{\phi(q)}\right| + O_{a,A}\left(\frac{x}{(\log x)^A}\right) \ll_A \frac{x}{(\log x)^A}.\end{aligned}$$

Putting all this together and using the triangle inequality, we get that the left hand side of (7) is

$$\begin{aligned}\le \sum_{\substack{R/2<r\le R\\(r,a)=1}} \Bigg| &\sum_{\substack{s\le\frac{RL}{r}\\(s,a)=1}} \frac{x}{\phi(sr)}\left(1-\frac{s}{RL/r}\right) - \sum_{\substack{s\le M\\(s,a)=1}} \frac{x}{\phi(sr)}\left(1-\frac{s}{M}\right) - \sum_{\substack{\frac{x}{RL}<q\le\frac{x}{rM}\\(q,a)=1}} \frac{x}{\phi(qr)}\\ &- x\left(\frac{C_1(a,r)}{r}\log(r'M)+\frac{C_3(a,r)}{r} - \sum_{\substack{s\le M\\(s,a)=1}} \frac{1}{\phi(sr)}\left(1-\frac{s}{M}\right)\right)\Bigg| + O_{a,A,D,\lambda}\left(\frac{x}{(\log x)^A}\right), \quad (8)\end{aligned}$$

since M is an integer. If M is not an integer, we have to add an error term of size

$$\ll x \sum_{R/2<r\le R} \frac{\log\log M}{\phi(r)M^2} \ll \frac{x\log\log M}{M^2}.$$

(We already used the fact that $x\sum_{R/2<r\le R}\frac{\log\log(RL/r)}{\phi(r)(RL/r)^2} \ll \frac{x\log\log L}{L^2}$ in (8).) Applying the triangle inequality once more gives that (8) is

$$\begin{aligned}
&\le x\sum_{\substack{R/2<r\le R\\(r,a)=1}}\left|\sum_{\substack{s\le\frac{RL}{r}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(1-\frac{s}{RL/r}\right)-\frac{C_1(a,r)}{r}\log\left(\frac{r'RL}{r}\right)-\frac{C_3(a,r)}{r}\right|\\
&\qquad + x\sum_{\substack{R/2<r\le R\\(r,a)=1}}\left|\sum_{\substack{\frac{x}{RL}<q\le\frac{x}{rM}\\(q,a)=1}}\frac{1}{\phi(qr)}-\frac{C_1(a,r)}{r}\log\left(\frac{RL}{rM}\right)\right|+O_{a,A,D,\lambda}\left(\frac{x}{(\log x)^A}\right),
\end{aligned}$$

which by Lemma 5.3 is

$$\begin{aligned}
&\ll_{a,A,D,\lambda} x\sum_{\substack{R/2<r\le R\\(r,a)=1}}\frac{3^{\omega(r)}\log(RL)}{RL}+x\sum_{\substack{R/2<r\le R\\(r,a)=1}}\frac{3^{\omega(r)}\log(x/RL)}{x/RL}+\frac{x}{(\log x)^A}\\
&\ll \frac{x(\log R)^2}{RL}+\frac{x}{(\log x)^A}\\
&\ll \frac{x}{(\log x)^A}.
\end{aligned}$$

□

Proof of Theorem 3.4. Taking $M=1$ in Proposition 6.1 and applying Lemma 5.3 and the triangle inequality, we get

$$\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{x}{r}\\(q,a)=1}}(\psi(x;qr,a)-\Lambda(a))-\frac{x}{r}\left(C_1(a,r)\log\left(\frac{(r')^2x}{er}\right)+2C_2(a,r)\right)\right|\ll_{a,A,\lambda}\frac{x}{\log^{A+1}x}.$$

Taking dyadic intervals, one can easily use this to show that the whole sum over $r\le R$ is $\ll_{a,A}\frac{x}{\log^A x}$. The result follows by exchanging the order of summation:

$$\begin{aligned}
\sum_{\substack{q\le\frac{x}{r}\\(q,a)=1}}\sum_{\substack{|a|<n\le x\\n\equiv a \bmod qr}}\Lambda(n) &= \sum_{\substack{|a|<n\le x\\n\equiv a \bmod r}}\Lambda(n)\sum_{\substack{q\le\frac{x}{r}:\\qr|n-a}}1\\
&= \sum_{\substack{|a|<n\le x\\n\equiv a \bmod r}}\Lambda(n)\tau\left(\frac{n-a}{r}\right).
\end{aligned}$$

(the last equality is exact if $a>0$, else we have to add a neglegible error term.)

□

Proof of Theorem 3.1. For the first result, we take $M(r,x) := M(x)$ in Proposition 6.1. By Lemma 5.4, we have that

$$\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}\left|\frac{\phi(a)}{a}\frac{x}{rM}\mu(a,r,M)-x\left(\frac{C_1(a,r)}{r}\log(r'M)+\frac{C_3(a,r)}{r}-\sum_{\substack{s\le M\\(s,a)=1}}\frac{1}{\phi(rs)}\left(1-\frac{s}{M}\right)\right)\right|$$

$$\le x\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}|E(a,r,M)|\ll_a\frac{x}{M^{\frac{205}{538}-\epsilon}}\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}\frac{\prod_{p|r}\left(1+\frac{1}{p^{\delta}}\right)}{r}\ll\frac{x}{M^{\frac{205}{538}-\epsilon}},\quad(9)$$

hence the result follows by the triangle inequality.

The second result is a bit more delicate, since we have the full range of r, and the innermost sum depends on R. For this reason, we need to go back to the proof of Proposition 6.1. We first split the sum over r into the two intervals $r\le R/(\log x)^B$ and $R/(\log x)^B<r\le R$, where we take $B=B(2A)$ as in Theorem 1.4, and we can assume that $B(2A)\ge 2A$. The first part of the sum is treated using this Theorem:

$$\sum_{\substack{r\le\frac{R}{(\log x)^B}\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)-\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,M)\right|$$

$$\ll_{a,A,\lambda}\frac{x}{(\log x)^{2A}}+\frac{x}{(\log x)^B},$$

since $\frac{R}{(\log x)^B}\cdot\frac{x}{RM}=\frac{x}{M(\log x)^B}\le\frac{x}{(\log x)^B}$. For the rest of the sum, we argue as in the proof of Proposition 6.1. We split the sum over q as follows:

$$\sum_{\substack{q\le\frac{x}{RM}\\(q,a)=1}}=\sum_{\substack{q\le\frac{x}{RL}\\(q,a)=1}}+\sum_{\substack{\frac{x}{RL}<q\le\frac{x}{r}\\(q,a)=1}}-\sum_{\substack{\frac{x}{RM}<q\le\frac{x}{r}\\(q,a)=1}}.$$

Taking P to be either $\frac{R}{r}L$ or $\frac{R}{r}M$, we have that $P\le L(\log x)^B$ (instead of $P\le 2L$). The rest of the proof goes through, and we get that

$$\sum_{\substack{\frac{R}{L}<r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)-x\left(\frac{C_1(a,r)}{r}\log\left(r'RM/r\right)+\frac{C_3(a,r)}{r}\right.\right.$$

$$\left.\left.-\sum_{\substack{s\le RM/r\\(s,a)=1}}\frac{1}{\phi(rs)}\left(1-\frac{s}{RM/r}\right)\right)\right|\ll_{a,A,D,\lambda}\frac{x}{(\log x)^{2A}}+E_2(x,M),\quad(10)$$

where $E_2(x,M)$ is the error coming from the fact that $\frac{R}{r}M$ is not an integer, which is

$$\ll x\sum_{\frac{R}{L}<r\le R}\frac{\log\log(RM/r)}{\phi(r)RM/r}\frac{1}{RM/r}\ll\frac{x}{(RM)^2}\sum_{\frac{R}{L}<r\le R}\frac{r^2\log\log(RM/r)}{\phi(r)}\ll\frac{x\log\log M}{M^2}.$$

We finish the proof by applying Lemma 5.4 and the triangle inequality. $\square$

Proof of Corollary 3.2. By the triangle inequality we have

$$\sum_{\substack{r\le R\\(r,a)=1}}\left|\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\right| \le \sum_{\substack{r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right.$$
$$\left. -\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\right| + \sum_{\substack{r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right|,$$

hence by Theorem 3.1 we get the lower bound

$$\sum_{\substack{r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right| \ge \frac{\phi(a)}{a}\frac{x}{RM}\sum_{\substack{r\le R\\(r,a)=1}}|\mu(a,r,RM/r)|$$
$$-O_\epsilon\left(\frac{x}{M^{\frac{743}{538}-\epsilon}}\right),$$

since for M large enough, $\mu(a,r,RM/r)\le 0$. For the upper bound, we write

$$\sum_{\substack{r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right| \le \sum_{\substack{r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le \frac{x}{RM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right.$$
$$\left.-\sum_{\substack{r\le R\\(r,a)=1}}\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,RM/r)\right| + \sum_{\substack{r\le R\\(r,a)=1}}\left|\frac{\phi(a)}{a}\frac{x}{RM}\mu(a,r,M)\right|$$
$$\le \frac{\phi(a)}{a}\frac{x}{RM}\sum_{\substack{r\le R\\(r,a)=1}}|\mu(a,r,RM/r)| + O_\epsilon\left(\frac{x}{M^{\frac{743}{538}-\epsilon}}\right).$$

The result follows by the definition of $\mu(a,r,RM/r)$. Note that if $a=\pm1$, then we have

$$2\sum_{\substack{r\le R\\(r,a)=1}}|\mu(a,r,RM/r)| = \sum_{r\le R}\left(\log(RM/r)+2C_5+\sum_{p\mid r}\frac{\log p}{p}\right)$$
$$=(R+O(1))\left(\log M+1+2C_5+O\left(\frac{\log R}{R}\right)\right)+\sum_{p\le R}\frac{\log p}{p}\left\lfloor\frac{R}{p}\right\rfloor,$$

by Stirling's approximation. The last sum can be handled without much effort:

$$\sum_{p\le R}\frac{\log p}{p}\left\lfloor\frac{R}{p}\right\rfloor = R\sum_{p\le R}\frac{\log p}{p^2}+O\left(\sum_{p\le R}\frac{\log p}{p}\right)$$
$$= R\left(\sum_{p}\frac{\log p}{p^2}+O\left(\frac{1}{R}\right)\right)+O\left(\log R\right).$$

Hence,

$$\sum_{\substack{r\le R\\(r,a)=1}}|\mu(a,r,RM/r)| = R\left(\frac{1}{2}\log M + C_6\right)+O(\log(RM)).$$

□

Proof of Proposition 3.5. Exchanging the order of summation as in the proof of Theorem 3.4, we get that

$$\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{x}{r}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)-x\left(\frac{C_1(a,r)}{r}\log r'+\frac{C_3(a,r)}{r}\right)\right|\ll\frac{x}{(\log x)^A}.$$

As we have seen in the proof of Proposition 6.1, we can give a good estimate for the part of the sum over q where $\frac{x}{RL}<q\le\frac{x}{r}$ by switching divisors and using the Bombieri-Vinogradov theorem (which explains the restriction on R). Doing so and applying Lemma 5.4, we get that

$$\sum_{\substack{\frac{R}{2}<r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{x}{RL}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right|\ll\frac{x}{L},$$

which concludes the proof. □

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